ADVANCED AUDIO SIGNAL PROCESSING

Sound source localization

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Modelization of the problem: *from acoustics to signal*

❑ Solution

- *pour **une source ponctuelle et impulsionnelle** :* **fonction de Green**

La solution de l'équation des ondes en $\vec{r}$ à l'instant t, notée $G(\vec{r},t,\overrightarrow{r_0},t_0)$ pour une source ponctuelle et impulsionnelle placée au point $\overrightarrow{r_0}$ et allumée à l'instant t_0, $\delta(\vec{r}-\overrightarrow{r_0})\delta(t-t_0)$, est appelée fonction de Green.
Elle vérifie :

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r},t,\overrightarrow{r_0},t_0) = \delta(\vec{r}-\overrightarrow{r_0})\delta(t-t_0)$$

- *pour **une source quelconque***

Connaître $G(\vec{r},t,\overrightarrow{r_0},t_0)$ permet de connaître le potentiel des vitesses en tout $\vec{r}$, à tout instant t, pour une source quelconque $s(\vec{r},t)$ connue, distribuée sur le domaine $\mathcal{D}$ et allumée aux instants de $I \subset \mathbb{R}_+$.
En effet :

$$\begin{array}{rcl} \delta(\vec{r}-\overrightarrow{r_0})\delta(t-t_0) & \rightarrow & G(\vec{r},t,\overrightarrow{r_0},t_0) \\ s(\vec{r},t) = \iiint_{\mathcal{D}} \int_{\mathbb{R}_+} s(\overrightarrow{r_0},t_0)\delta(\vec{r}-\overrightarrow{r_0})\delta(t-t_0)dt_0\, d\overrightarrow{r_0} & \rightarrow & \phi(\vec{r},t) = \iiint_{\mathcal{D}} \int_{\mathbb{R}_+} s(\overrightarrow{r_0},t_0)G(\vec{r},t,\overrightarrow{r_0},t_0)dt_0\, d\overrightarrow{r_0} \end{array}$$

- *Calcul de la fonction de Green en 3D*

Méthode :
a- Transformée de Fourier spatiale et temporelle de l'équation de d'Alembert;
b- Expression de la TF de $G(\vec{r},t,\overrightarrow{r_0},t_0)$, notée $\tilde{G}(\vec{\chi},\omega)$
c- TF inverse de $\tilde{G}(\vec{\chi},\omega)$ par intégration sur les variables $\vec{\chi}$ et ω.

$$\Rightarrow G(\vec{r},t,\overrightarrow{r_0},t_0) = \frac{-1}{4\pi\|\vec{r}-\overrightarrow{r_0}\|}\,\delta\left(\frac{\|\vec{r}-\overrightarrow{r_0}\|}{c} - (t-t_0)\right)$$

Et donc $\phi(\vec{r},t) = \frac{-1}{4\pi}\iiint_{\mathcal{D}} \frac{1}{\|\vec{r}-\overrightarrow{r_0}\|}\, s\left(\overrightarrow{r_0}, t - \frac{\|\vec{r}-\overrightarrow{r_0}\|}{c}\right) d\overrightarrow{r_0}$

Notations: back to acoustics considerations

You saw during course 2 by H. Boutin that the velocity potential Φ at position r can be expressed by

$$\Phi(\mathrm{r}, t) = -\frac{1}{4\pi} \iiint_{\mathcal{D}} \frac{1}{\|\mathrm{r} - \mathrm{r}_0\|} s\left(\mathrm{r}_0, t - \frac{\|\mathrm{r} - \mathrm{r}_0\|}{c}\right) \mathrm{d}\mathrm{r}_0 \tag{1}$$

Lets now consider a punctual source, positioned at r^{S}, such that $s(\mathrm{r}_0, t) = \delta(\mathrm{r}_0 - \mathrm{r}^{\mathrm{S}})s(t)$. Then, one have

$$\begin{aligned}
\Phi(\mathrm{r}, t) &= -\frac{1}{4\pi} \iiint_{\mathcal{D}} \frac{1}{\|\mathrm{r} - \mathrm{r}_0\|} \delta(\mathrm{r}_0 - \mathrm{r}^{\mathrm{S}}) s\left(t - \frac{\|\mathrm{r} - \mathrm{r}_0\|}{c}\right) \mathrm{d}\mathrm{r}_0 \\
&= -\frac{1}{4\pi} \iiint_{\mathcal{D}} \frac{1}{\|\mathrm{r} - \mathrm{r}^{\mathrm{S}}\|} \delta(\mathrm{r}_0 - \mathrm{r}^{\mathrm{S}}) s\left(t - \frac{\|\mathrm{r} - \mathrm{r}^{\mathrm{S}}\|}{c}\right) \mathrm{d}\mathrm{r}_0 \\
&= -\frac{1}{4\pi} \frac{1}{\|\mathrm{r} - \mathrm{r}^{\mathrm{S}}\|} s\left(t - \frac{\|\mathrm{r} - \mathrm{r}^{\mathrm{S}}\|}{c}\right) = \Phi(\mathrm{r}, \mathrm{r}^{\mathrm{S}}, t)
\end{aligned}$$

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&= -\frac{1}{4\pi}\frac{1}{\|\mathrm{r}-\mathrm{r}^{\mathrm{s}}\|} s\left(t-\frac{\|\mathrm{r}-\mathrm{r}^{\mathrm{s}}\|}{c}\right) = \Phi(\mathrm{r},\mathrm{r}^{\mathrm{s}},t)
\end{aligned}$$

For a purely monochromatic source with $s(t) = e^{jkct}$, this writes as

$$\Phi(\mathrm{r},\mathrm{r}^{\mathrm{s}},t) = -\frac{1}{4\pi\|\mathrm{r}-\mathrm{r}^{\mathrm{s}}\|} e^{-jk\|\mathrm{r}-\mathrm{r}^{\mathrm{s}}\|} e^{jkct}. \tag{2}$$

Notations: back to acoustics considerations

But a microphone placed at r is not sensitive to the velocity potential Φ, but rather to pressure variations p, so that

$$m(\mathrm{r}, \mathrm{r}^{\mathrm{s}}, t) = S\ a(t) * p(\mathrm{r}, \mathrm{r}^{\mathrm{s}}, t), \tag{3}$$

with S the microphone sensitivity (in V/Pa) and $a(t)$ the impulse response of the microphone, such that $a(t) = TF^{-1}[A(f)]$, with $A(f)$ the microphone frequency response.

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If we recall that

$$p(\mathrm{r}, \mathrm{r}^{\mathrm{s}}, t) = -\rho_0 \frac{\partial \Phi(\mathrm{r}, \mathrm{r}^{\mathrm{s}}, t)}{\partial t}, \tag{4}$$

then the pressure measured by the microphone at r for a monochromatic source at r^{s} can be expressed as

$$p(\mathrm{r}, \mathrm{r}^{\mathrm{s}}, t) = \frac{j\rho_0 kc}{4\pi \|\mathrm{r} - \mathrm{r}^{\mathrm{s}}\|} e^{-jk\|\mathrm{r}-\mathrm{r}^{\mathrm{s}}\|} e^{jkct}. \tag{5}$$

All is relative . . .

We often take the pressure $p_0(\mathrm{r}^{\mathrm{S}}, t)$ at the center O of the frame as a reference, with

$$p_0(\mathrm{r}^{\mathrm{S}}, t) = p(0, \mathrm{r}^{\mathrm{S}}, t) = \frac{j\rho_0 kc}{4\pi \|\mathrm{r}^{\mathrm{S}}\|} e^{-jk\|\mathrm{r}^{\mathrm{S}}\|} e^{jkct}. \tag{6}$$

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Then, the pressure measured by a microphone at position r and a source emitting from position r^{S} can be expressed along

$$p(\mathrm{r}, \mathrm{r}^{\mathrm{S}}, t) = \frac{\|\mathrm{r}^{\mathrm{S}}\|}{\|\mathrm{r} - \mathrm{r}^{\mathrm{S}}\|} \; e^{jk\|\mathrm{r}^{\mathrm{S}}\|} \; e^{-jk\|\mathrm{r}-\mathrm{r}^{\mathrm{S}}\|} \; p_0(\mathrm{r}^{\mathrm{S}}, t). \tag{7}$$

All is relative . . .

Then, coming back to the expression of the output signal $m(\mathrm{r}, \mathrm{r}^{\mathrm{S}}, t)$, and with the hypothesis $a(t) = \delta(t)$ (i.e. the microphone exactly reproduces the pression p up to a constant, that is $\mathrm{A}(\mathrm{f}) = 1$), then one gets

$$m(\mathrm{r}, \mathrm{r}^{\mathrm{S}}, t) = \frac{\|\mathrm{r}^{\mathrm{S}}\|}{\|\mathrm{r} - \mathrm{r}^{\mathrm{S}}\|} \, e^{jk\|\mathrm{r}^{\mathrm{S}}\|} \, e^{-jk\|\mathrm{r}-\mathrm{r}^{\mathrm{S}}\|} \, m_0(\mathrm{r}^{\mathrm{S}}, t), \tag{6}$$

where $m_0(\mathrm{r}^{\mathrm{S}}, t) = Sp_0(\mathrm{r}^{\mathrm{S}}, t)$ represents the *virtual* signal at the origin 0.

Since everything is supposed monochromatic, this virtual signal can be simply written as

$$m_0(\mathrm{r}^{\mathrm{S}}, t) = Ke^{jkct} = s^0(t), \tag{7}$$

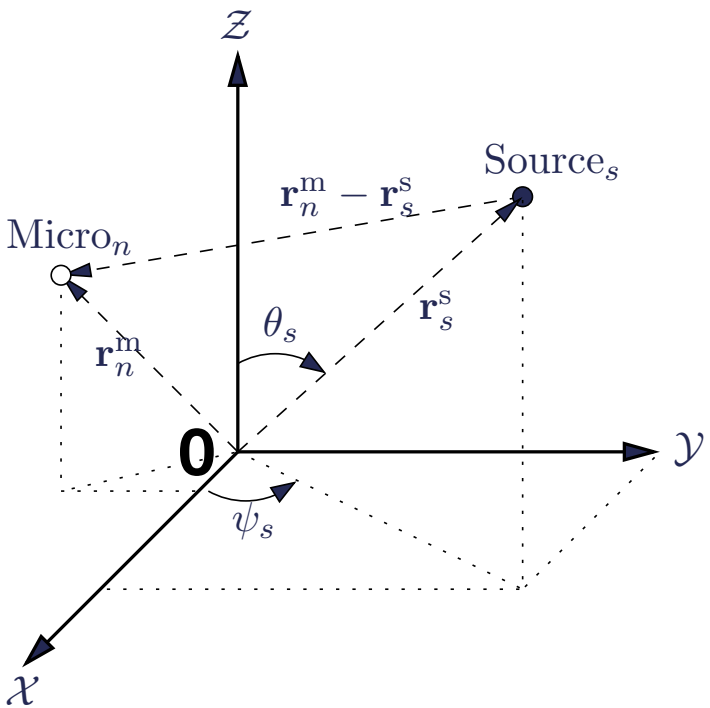
with $K = \frac{jS\rho_0 kc}{4\pi\|\mathrm{r}^{\mathrm{S}}\|} e^{-jk\|\mathrm{r}^{\mathrm{S}}\|}$ a constant.

$s^0(t)$ represents the relative contribution of the monochromatic source at position 0. It will be now considered as the reference signal.

Notations

In all the following, we now consider :

- S **punctual** monochromatic sound sources, placed at positions $\mathrm{r}_1^{\mathrm{s}}, \mathrm{r}_2^{\mathrm{s}}, \ldots, \mathrm{r}_S^{\mathrm{s}}$
- N **omnidirectional** microphones with identical frequency response $A(f)$, placed at positions $\mathrm{r}_1^{\mathrm{m}}, \mathrm{r}_2^{\mathrm{m}}, \ldots, \mathrm{r}_N^{\mathrm{m}}$.



- s^{th} source position: $\mathrm{r}_s^{\mathrm{s}}$
- n^{th} microphone position: $\mathrm{r}_n^{\mathrm{m}}$
- All microphones have a frequency response $A(f) = 1$

Notations

Then the signal outputed by the n^{th} microphone at position $\mathrm{r}_n^{\mathrm{m}}$ can be written as

$$m_n(t) = m(\mathrm{r}_n^{\mathrm{m}}, t) = \sum_{s=1}^{S} m(\mathrm{r}_n^{\mathrm{m}}, \mathrm{r}_s^{\mathrm{s}}, t) \tag{8}$$

This can be rewritten as

$$m_n(t) = \sum_{s=1}^{S} \frac{\|\mathrm{r}^{\mathrm{s}}\|}{\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}^{\mathrm{s}}\|} \; e^{jk\|\mathrm{r}^{\mathrm{s}}\|} \; e^{-jk\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}^{\mathrm{s}}\|} \; m_0(\mathrm{r}^{\mathrm{s}}, t) \tag{9}$$

$$= \sum_{s=1}^{S} \frac{\|\mathrm{r}^{\mathrm{s}}\|}{\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}^{\mathrm{s}}\|} \; e^{jk\|\mathrm{r}^{\mathrm{s}}\|} \; e^{-jk\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}^{\mathrm{s}}\|} \; s_s^0(t) \tag{10}$$

$$= \sum_{s=1}^{S} \alpha_n(\mathrm{r}_s^{\mathrm{s}}) s_s^0(t - \tau_n(\mathrm{r}_s^{\mathrm{s}})) \tag{11}$$

where $s_s^0(t)$ is the contribution of the s^{th} monochromatic source at the frame origin $\mathbf{0}$.

Notations

More generally, one writes

$$m_n(t) = \sum_{s=1}^{S} \alpha_n(\mathrm{r}_s^{\mathrm{s}}) s_s^0(t - \tau_n(\mathrm{r}_s^{\mathrm{s}})) + b_n(t), \tag{12}$$

with:

- $\alpha_n(\mathrm{r}) = \frac{\|\mathrm{r}\|}{\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}\|}$ the relative attenuation caused by the wave propagation;
- $\tau_n(\mathrm{r}) = \frac{\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}\|}{c} - \frac{\|\mathrm{r}\|}{c}$ the relative delay caused by the wave propagation;
- $b_n(t)$ some additive noise present on the n^{th} microphone.

And again . . . all is relative to the virtual signal "received" at the origin 0.

Notations

And everything can also be written in the frequency domain, along

$$M_n(k) = \sum_{s=1}^{S} V_n(\mathrm{r}_s^{\mathrm{s}}, k) S_s^0(k) + B_n(k), \tag{13}$$

where

$$V_n(\mathrm{r}, k) = \|\mathrm{r}\| e^{jk\|\mathrm{r}\|} \frac{e^{-jk\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}\|}}{\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}\|} \tag{14}$$

represents some kind of *relative frequency response* between the monochromatic signal of frequency k emitted by a source at r and the output of a microphone placed at $\mathrm{r}_n^{\mathrm{m}}$.

Notations: conclusion

Everything can be summed up into some vectorial notations ($.^T$ denotes the transposition operation):

- $\mathrm{M}(k) = (M_1(k), \ldots, M_N(k))^T$: the observation vector, of size $(N \times 1)$
- $\mathrm{V}(\mathrm{r}, k) = (V_1(\mathrm{r}, k), \ldots, V_N(\mathrm{r}, k))^T$: the **steering vector**, of size $(N \times 1)$
- $\mathrm{S}^0(k) = (S_1^0(k), \ldots, S_S^0(k))$: the source vector, of size $(S \times 1)$
- $\mathrm{B}(k) = (B_1(k), \ldots, B_N(k))^T$: the noise vector, of size $(N \times 1)$

Then, all sources/microphones can be regrouped inside one matrix:

- $\mathcal{V}(k) = (\mathrm{V}(\mathrm{r}_1^{\mathrm{s}}, k), \ldots, \mathrm{V}(\mathrm{r}_S^{\mathrm{s}}, k))$: the **steering matrix**, of size $(N \times S)$,

so that the previous equation can be rewritten as

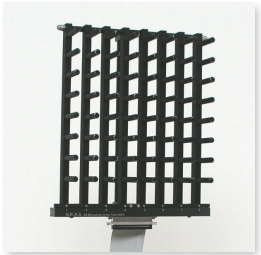
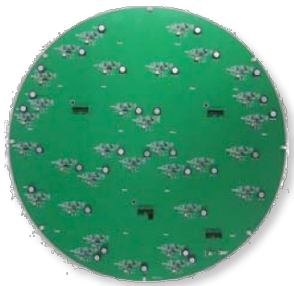
$$\mathrm{M}(k) = \mathcal{V}(k)\mathrm{S}^0(k) + \mathrm{B}(k) \tag{15}$$

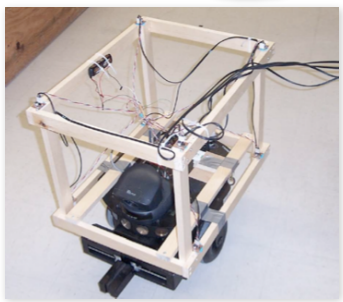
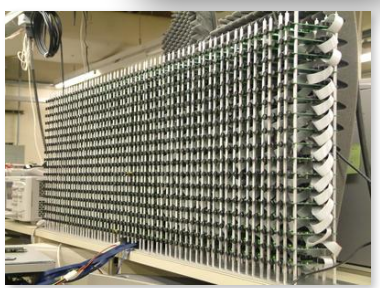
which constitutes the fundamental relation linking the S sources to the N microphones outputs.

Array processing: an example

What's a microphone array?

Some examples:

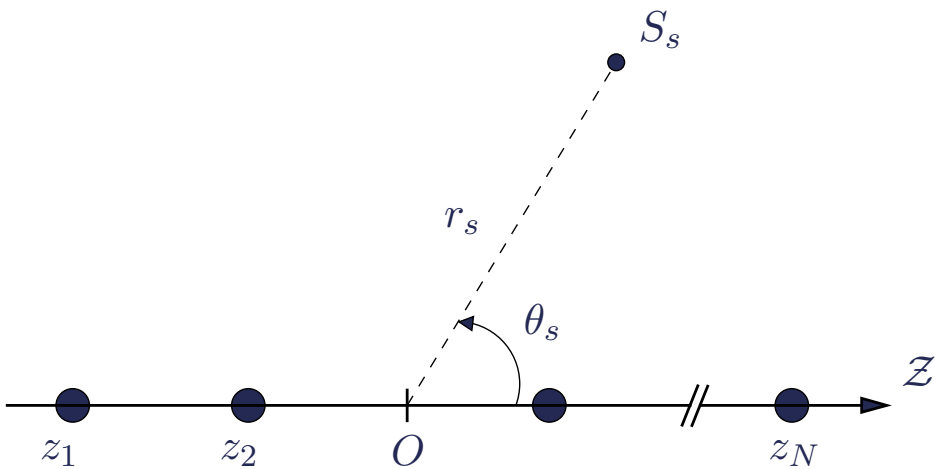




A microphone array is a fixed, geometrical, arrangement of identical (often calibrated) microphones.

Example: the linear array

Lets now consider a linear array, made of N microphones placed along a line $\mathcal{Z}$ and at positions z_n.



Taking 0 as the frame origin, a source position $\mathrm{r}_s^{\mathrm{s}}$ in space is then characterized by the vector $\mathrm{r}_s^{\mathrm{s}} = (r_s, \theta_s)$.

Consequently, one can write $\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}_s^{\mathrm{s}}\| = \sqrt{r_s^2 + z_n^2 - 2 r_s z_n \cos\theta_s}$

Example: the linear array

Consequently, one component of the steering vector, which writes as

$$V_n(\mathrm{r}, k) = \|\mathrm{r}\| e^{jk\|\mathrm{r}\|} \frac{e^{-jk\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}\|}}{\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}\|}$$

. . . can be "simplified" along

$$V_n(\mathrm{r}, k) = V_n(r, \theta, k) = \frac{re^{jkr}}{\sqrt{r^2 + z_n^2 - 2rz_n \cos\theta_s}} e^{-jk\sqrt{r^2 + z_n^2 - 2rz_n \cos\theta_s}}.$$

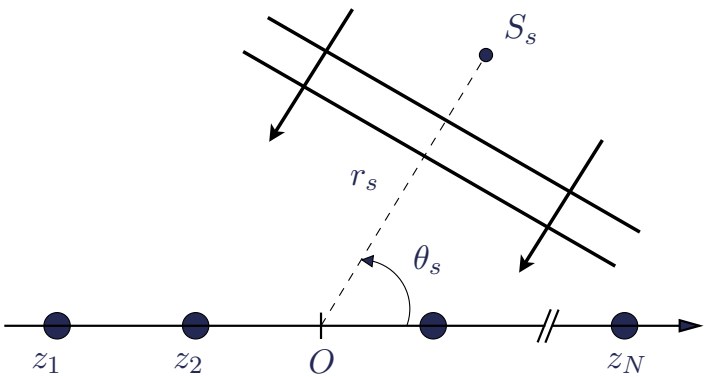
Now, if one supposes that the source is sufficiently far from the microphones (in practice, this means that $r \to +\infty$), then one gets $V_n^{\infty}(\theta, k)$, with

$$V_n^{\infty}(\theta, k) = e^{-jkz_n \cos\theta} \tag{16}$$

How can we interpret this simple equation?

Example: the linear array

Taking $r_s \to +\infty$ is the same as considering a planar wave propagation.



The source/microphone transfer relation is given by

$$V_n^{\infty}(\theta, k) = e^{-jkz_n \cos\theta}$$

. . . which represents a **pure delay**!

It then means that one can write:

$$m_n^{\infty}(t) = \sum_{s=1}^{S} s_s^0(t - \tau_n^{\infty}(\theta_s))(+b_n(t)), \tag{17}$$

with

$$\tau_n^{\infty}(\theta) = \frac{z_n}{c}\cos\theta \tag{18}$$

The signals among all microphones only differ from a pure delay $\tau_n^{\infty}(\theta_s)$ that depends on the angular source position θ_s.

To sum up

In the general case, one has:

$$m_n(t) = \sum_{s=1}^{S} \alpha_n(\mathrm{r}_s^{\mathrm{s}}) s_s^0(t - \tau_n(\mathrm{r}_s^{\mathrm{s}})) + b_n(t), \tag{19}$$

which can be rewritten in the frequency domain as

$$M_n(k) = \sum_{s=1}^{S} V_n(\mathrm{r}_s^{\mathrm{s}}, k) S_s^0(k) + B_n(k), \tag{20}$$

with

$$V_n(\mathrm{r}, k) = \|\mathrm{r}\| e^{jk\|\mathrm{r}\|} \frac{e^{-jk\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}\|}}{\|\mathrm{r}_n^{\mathrm{m}} - \mathrm{r}\|}. \tag{21}$$

For a linear array with planar wave fronts:

$$V_n^{\infty}(\theta, k) = e^{-jkz_n \cos\theta} \tag{22}$$

TDOA estimation through correlation for sound source localization

Principle

Idea: exploit the Time Delay Of Arrival (TDOA) between each signal of a microphones array.

$$m_n(t) = \sum_{s=1}^{S} \alpha_n(\mathrm{r}_s^{\mathrm{S}}) s_s^0 \left(t - \tau_n(\mathrm{r}_s^{\mathrm{S}})\right) + b_n(t). \tag{23}$$

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Practical implementation for localization: relies on 2 successive steps:

1. first, estimation $\widehat{\Delta T_{ij}}$ of the true TDOAs $\Delta T_{ij}(\mathrm{r}_s^{\mathrm{S}}) = \tau_i(\mathrm{r}_s^{\mathrm{S}}) - \tau_j(\mathrm{r}_s^{\mathrm{S}})$ between the i^{th} and j^{th} microphone of the array;
2. and then, exploitation of these delays to actually estimate the source position $\widehat{\mathrm{r}_s^{\mathrm{S}}}$, with the hope that $\widehat{\mathrm{r}_s^{\mathrm{S}}} = \mathrm{r}_s^{\mathrm{S}}$.

Towards some TDOA estimation

You have seen during your "random signal processing" course that a delay between two signal can be estimated through the cross-correlation $R_{ij}(\tau)$:

$$R_{ij}(\tau) = \mathbb{E}\left[m_i(t)m_j(t-\tau)\right]. \tag{24}$$

Since the available signal for the n^{th} microphone writes as

$$m_n(t) = \sum_{s=1}^{S} \alpha_n(\mathrm{r}_s^{\mathrm{s}}) s_s^0(t - \tau_n(\mathrm{r}_s^{\mathrm{s}})) + b_n(t), \tag{25}$$

then for one source in the environment, $R_{ij}(\tau)$ can be written as

[1] with the hypothesis that the source signal s is a wide-sense stationary process for which $R_{ss}(\tau) \leq R_{ss}(0)$.

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$$R_{ij}(\tau) = \alpha_i \alpha_j R_{ss}(\tau - \Delta T_{ij}) + R_{b_i b_j}(\tau), \tag{26}$$

where R_{ss} represents the source autocorrelation, and $R_{b_i b_j}$ the noise cross-correlation.

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Then $\Delta T_{ij} = \tau_i(\mathrm{r}_s^{\mathrm{s}}) - \tau_j(\mathrm{r}_s^{\mathrm{s}})$ can be estimated by detecting the position of the maximum[1] in $R_{ij}(\tau)$, i.e.

$$\widehat{\Delta T_{ij}} = \arg_\tau \max R_{ij}(\tau) \tag{27}$$

[1] with the hypothesis that the source signal s is a wide-sense stationary process for which $R_{ss}(\tau) \leq R_{ss}(0)$.

Estimation of the cross-correlation function

In practice, only an estimation $\widehat{R}_{ij}(\tau)$ of the cross-correlation $R_{ij}(\tau)$ can be obtained from a unique realization of the two signals $m_i(t)$ and $m_j(t)$ on a finite time window $[-T/2; T/2]$ of length T, i.e.

$$\widehat{R}_{ij}(\tau) = \frac{1}{T} \int_{-\infty}^{+\infty} m_{1,T}(t) m_{2,T}(t-\tau) \mathrm{d}t, \tag{28}$$

One can show that this estimator is a biased estimator of the cross-correlation $R_{ij}(\tau)$.

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One can show that this estimator is a biased estimator of the cross-correlation $R_{ij}(\tau)$.

How to choose the length T?
If we consider a planar wave propagation, we have shown that $\tau_n(\mathrm{r}_s^{\mathrm{S}}) = \tau_n^{\infty}(\theta_s) = \frac{z_n}{c}\cos\theta_s$, then one have

$$\Delta T_{ij} = \tau_i(\mathrm{r}_s^{\mathrm{S}}) - \tau_j(\mathrm{r}_s^{\mathrm{S}}) = \tau_i^{\infty}(\theta_s) - \tau_j^{\infty}(\theta_s) = \frac{z_i - z_j}{c}\cos\theta_s, \tag{29}$$

so that the maximal delay between two microphones is $D_{\max} = \frac{\max z_i - z_j}{c}$.

$\rightarrow$ *Then $\widehat{R}_{ij}(\tau)$ is shown to have a small variance and bias if $T >> D_{max}$.*

Estimation of the cross-correlation function

In practice we often compute the cross-correlation in the frequency domain, along

$$R_{ij}(\tau) = \int_{-\infty}^{+\infty} S_{m_i m_j}(f) e^{j2\pi f \tau} \mathrm{d}f, \tag{30}$$

where $S_{m_i m_j}(f)$ is the cross-power spectral density of the two signals m_i and m_j.

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Again, all these quantities must be estimated from a finite time observation of length T, so that one works in the end with

Generalized cross-correlation

$$\widehat{R}_{ij}(\tau) = \int_{-\infty}^{+\infty} \Psi(f) \widehat{S}_{m_{i,T} m_{j,T}}(f) e^{j2\pi f \tau} \mathrm{d}f, \tag{31}$$

with $\Psi(f)$ **a frequency weight function**, and $\widehat{S}_{m_{i,T} m_{j,T}}(f)$ the estimated cross-power spectral density of one realization of the two signals $m_{i,T}$ and $m_{j,T}$ (e.g. by a mean cross-periodogram approach).

Generalized cross-correlation: the PHAse Transform weight

As an example, a frequently used weight function is given by

$$\Psi^{\text{PHAT}}(f) = \frac{1}{\|S_{m_i m_j}(f)\|}, \tag{32}$$

which defines the Cross-power Spectrum Phase (CSP) or PHAse Transform (PHAT).

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Interest:
If $M_1(f) = \alpha_1 S(f) e^{-j2\pi f \tau_1}$ and $M_2(f) = \alpha_2 S(f) e^{-j2\pi f \tau_2}$, then

$$S_{m_1 m_2}(f) = M_1(f) M_2^*(f) = \alpha_1 \alpha_2 |S(f)|^2 e^{-j2\pi f(\tau_1 - \tau_2)} \tag{33}$$

$$\Psi^{\text{PHAT}}(f) = \frac{1}{\alpha_1 \alpha_2 |S(f)|^2} \tag{34}$$

Then:

$$\begin{aligned} R_{ij}^{\text{PHAT}}(\tau) &= \int_{-\infty}^{+\infty} \Psi^{\text{PHAT}}(f) S_{m_i m_j}(f) e^{j2\pi f \tau} \mathrm{d}f = \int_{-\infty}^{+\infty} e^{-j2\pi f(\tau_1 - \tau_2)} e^{j2\pi f \tau} \mathrm{d}f \\ &= \delta\left(\tau - (\tau_1 - \tau_2)\right). \end{aligned} \tag{35}$$

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One then have:

$$\begin{aligned} R_{ij}^{\text{PHAT}}(\tau) &= \delta\left(\tau - (\tau_1 - \tau_2)\right), \\ &= \delta\left(\tau - \Delta T_{ij}\right), \end{aligned} \tag{36}$$

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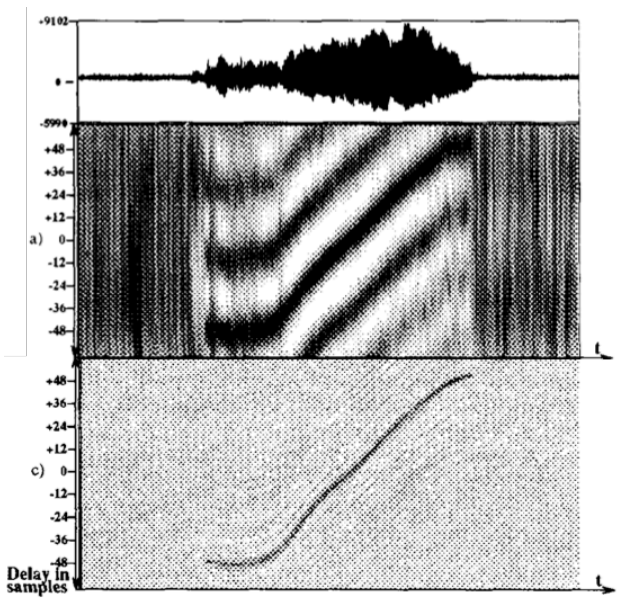
Example[2]:



$\rightarrow$ Signal

$\rightarrow$ "Traditional" cross-correlation

$\rightarrow$ CSP/PHAT

[2] Taken from (Omologo and Svaizer, 1994).

Generalized cross-correlation: the PHAse Transform weight

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Lets now do some code!

Generalized cross-correlation

Other weighting functions are proposed in the literature:

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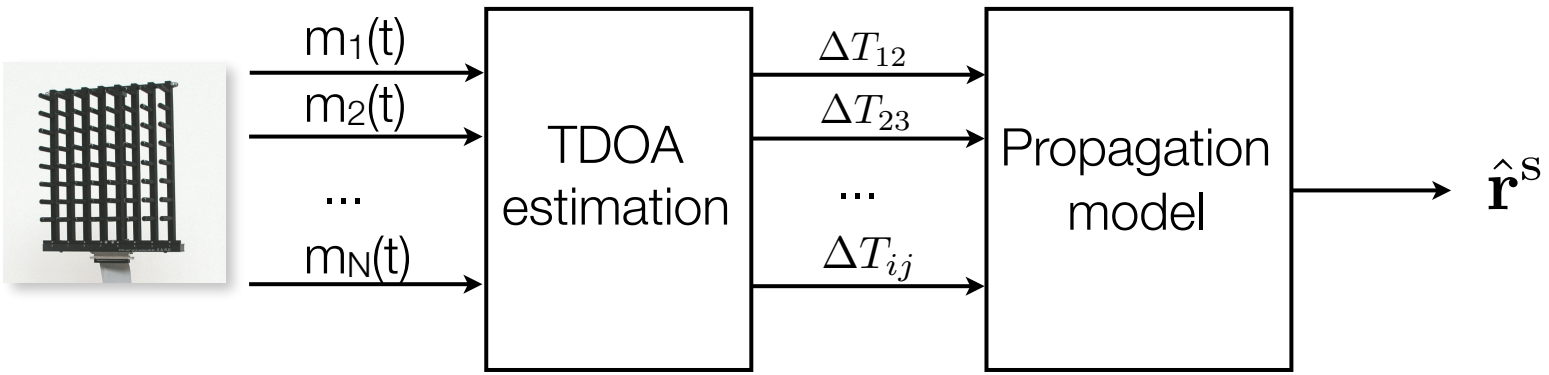
- ...and others ad-hoc functions, like the Reliability-Weighted PHAT (Valin et al., 2003), etc.

Most of them tends to increase/decrease the contribution of certain frequencies (on the basis in their high/low SNR for instance) to the overall cross-correlation (Portello, 2013).

From TDOA to source position

Recall that we have a 2-step approach:



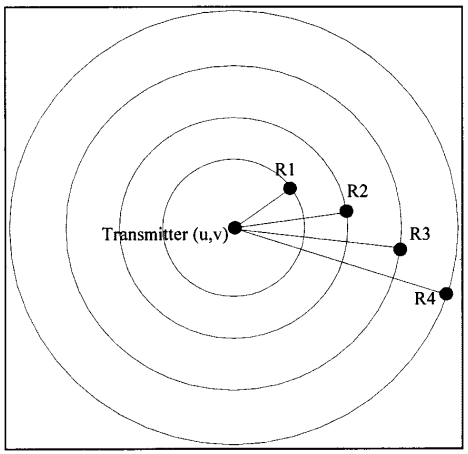
What's a "propagation model"?

It relates the estimated TDOA to the source position thanks to the array geometry.

For instance: for only 2 microphones spaced by d in the farfield, one have

$$\Delta T_{12} = \frac{d}{c} \sin\theta_s \Longrightarrow \widehat{\theta_s} = \sin^{-1}\left(\frac{c\widehat{\Delta T_{12}}}{d}\right) \tag{37}$$

From TDOA to source position



We only focus in the TDOAs $\Delta T_{1i} = \tau_1 - \tau_i$.

For a spherical propagation, the wave fronts write as, for all $i \in [1, \ldots, N]$

$$(x_i^{\mathrm{m}} - x^{\mathrm{s}})^2 + (y_i^{\mathrm{m}} - y^{\mathrm{s}})^2 + (z_i^{\mathrm{m}} - z^{\mathrm{s}})^2 = (d + c\Delta T_{1i})^2,$$

with d the distance between $R1$ and the source.

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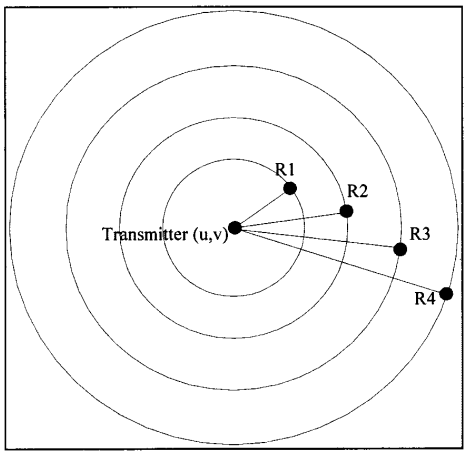

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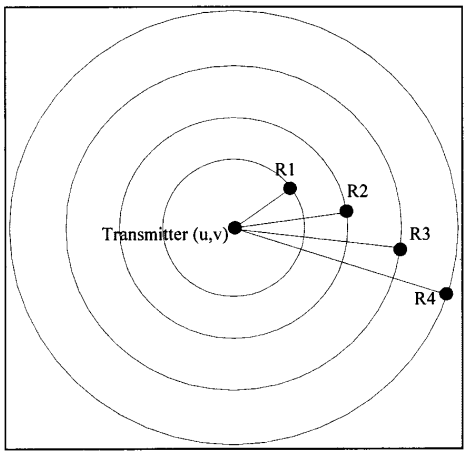
with d the distance between $R1$ and the source.

This equation can be rewritten with matrices, along

$$\begin{pmatrix} 2(x_1^{\mathrm{m}} - x_2^{\mathrm{m}}) & 2(y_1^{\mathrm{m}} - y_2^{\mathrm{m}}) & 2(z_1^{\mathrm{m}} - z_2^{\mathrm{m}}) & -2c\Delta T_{12} \\ 2(x_1^{\mathrm{m}} - x_3^{\mathrm{m}}) & 2(y_1^{\mathrm{m}} - y_3^{\mathrm{m}}) & 2(z_1^{\mathrm{m}} - z_3^{\mathrm{m}}) & -2c\Delta T_{13} \\ \vdots & \vdots & \vdots & \vdots \\ 2(x_1^{\mathrm{m}} - x_N^{\mathrm{m}}) & 2(y_1^{\mathrm{m}} - y_N^{\mathrm{m}}) & 2(z_1^{\mathrm{m}} - z_N^{\mathrm{m}}) & -2c\Delta T_{1N} \end{pmatrix} \begin{pmatrix} x^{\mathrm{s}} \\ y^{\mathrm{s}} \\ z^{\mathrm{s}} \\ d \end{pmatrix} =$$

$$\begin{pmatrix} c^2\Delta T_{12}^2 + (x_1^{\mathrm{m}})^2 + (y_1^{\mathrm{m}})^2 + (z_1^{\mathrm{m}})^2 - (x_2^{\mathrm{m}})^2 - (y_2^{\mathrm{m}})^2 - (z_2^{\mathrm{m}})^2 \\ c^2\Delta T_{13}^2 + (x_1^{\mathrm{m}})^2 + (y_1^{\mathrm{m}})^2 + (z_1^{\mathrm{m}})^2 - (x_3^{\mathrm{m}})^2 - (y_3^{\mathrm{m}})^2 - (z_3^{\mathrm{m}})^2 \\ \vdots \\ c^2\Delta T_{1N}^2 + (x_1^{\mathrm{m}})^2 + (y_1^{\mathrm{m}})^2 + (z_1^{\mathrm{m}})^2 - (x_N^{\mathrm{m}})^2 - (y_N^{\mathrm{m}})^2 - (z_N^{\mathrm{m}})^2 \end{pmatrix} \quad (38)$$

From TDOA to source position



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but ...

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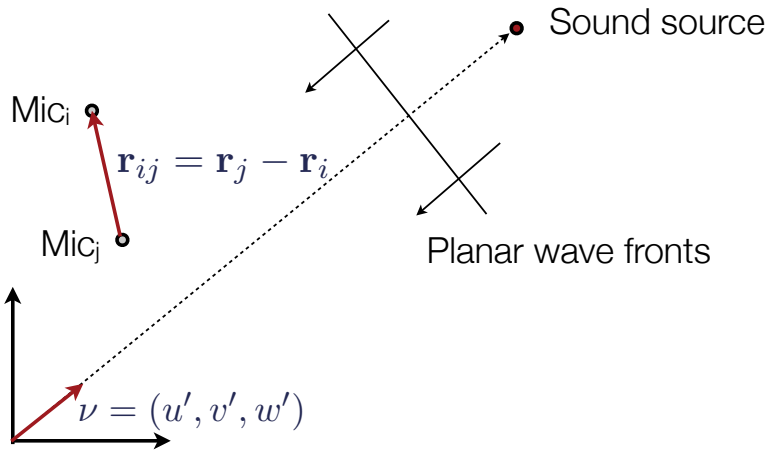
has to be (pseudo-)inversed, and its conditioning actually depends on ΔT_{1i} ...

From TDOA to source position

Instead, the following approach might be preferred:



The wave fronts being supposed planar, we can write

$$\boldsymbol{\nu}.\mathrm{r}_{i1}^{\mathrm{m}} = c\Delta T_{i1}, \qquad (39)$$

with $\boldsymbol{\nu} = (u, v, w)$ the vector pointing towards the source direction and . the scalar product.

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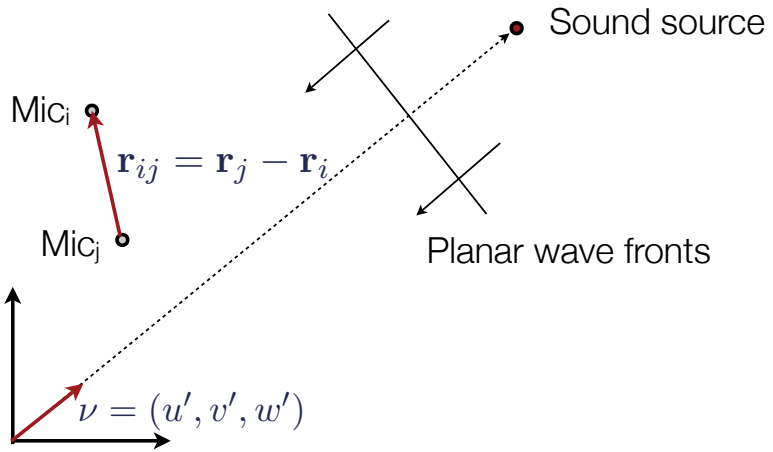

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This can be also written, for all microphones,

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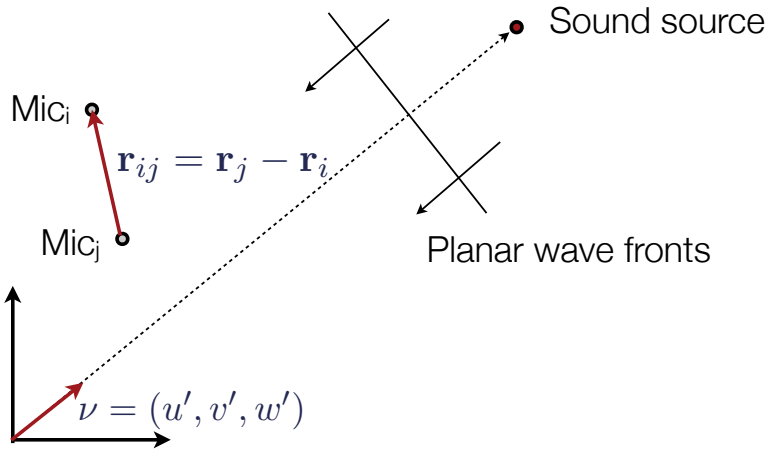

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. . . which now only depends on the microphone position/geometry!

Correlation techniques: conclusion

Pros:

- relatively "simple" approach, only relying a cross-correlation computation
- well-chosen pre-filtering functions can better the delay estimation performances

Cons:

- moving from the TDOAs to the estimated source position might be tricky (e.g. how to deal with scattering elements in the array?)
- poor performances of correlation strategies in the presence of noise, with low SNR conditions . . .
- . . . or in the presence of reverberations, which should then carefully taken into account (strategies not discussed here)

Beamforming approaches to localization

Principle

Basic idea:

Polarization of a microphones array towards a specific direction so as to

- amplify the signal coming from a given direction of interest,
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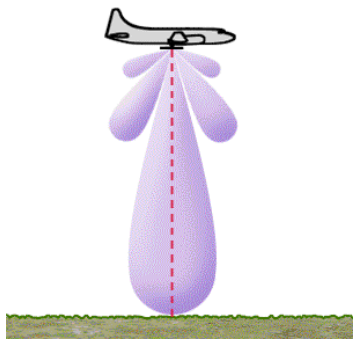
Applications:

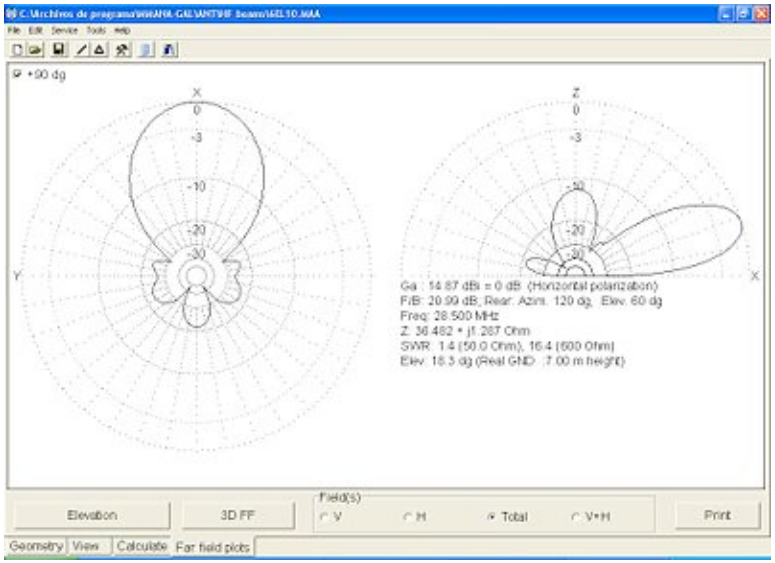
Telecommunications

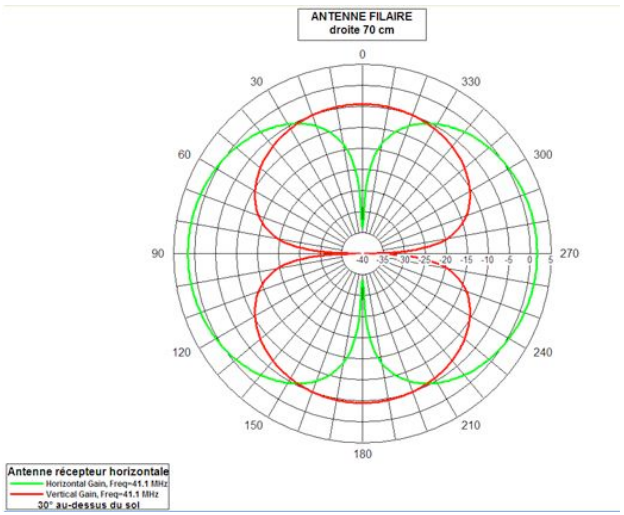


Radars

Principle

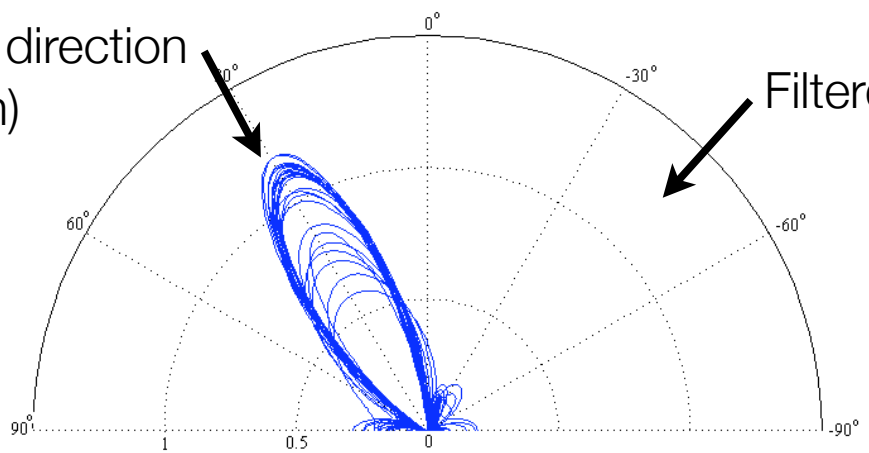
Some beampatterns:





Priviledged direction (polarization)

Filtered directions

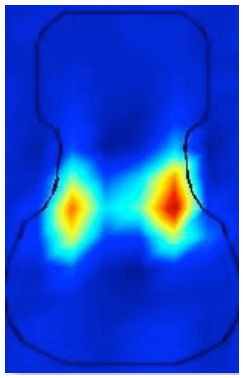


Principle

And what for?

- **Localization**

The array is successively polarized in every candidate positions. For each of them, the signal level can be computed and reported on an energy map.



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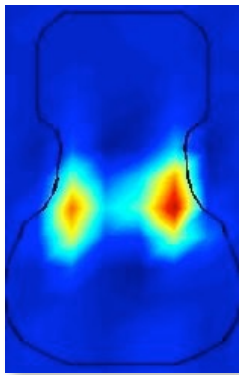
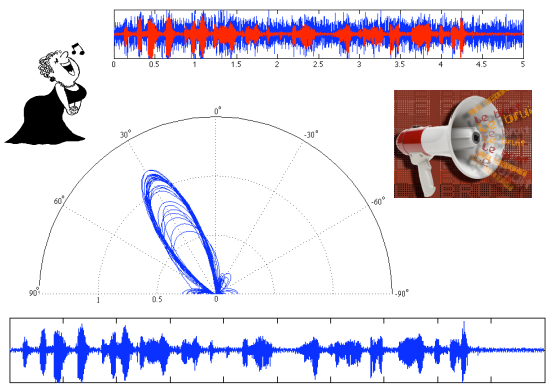


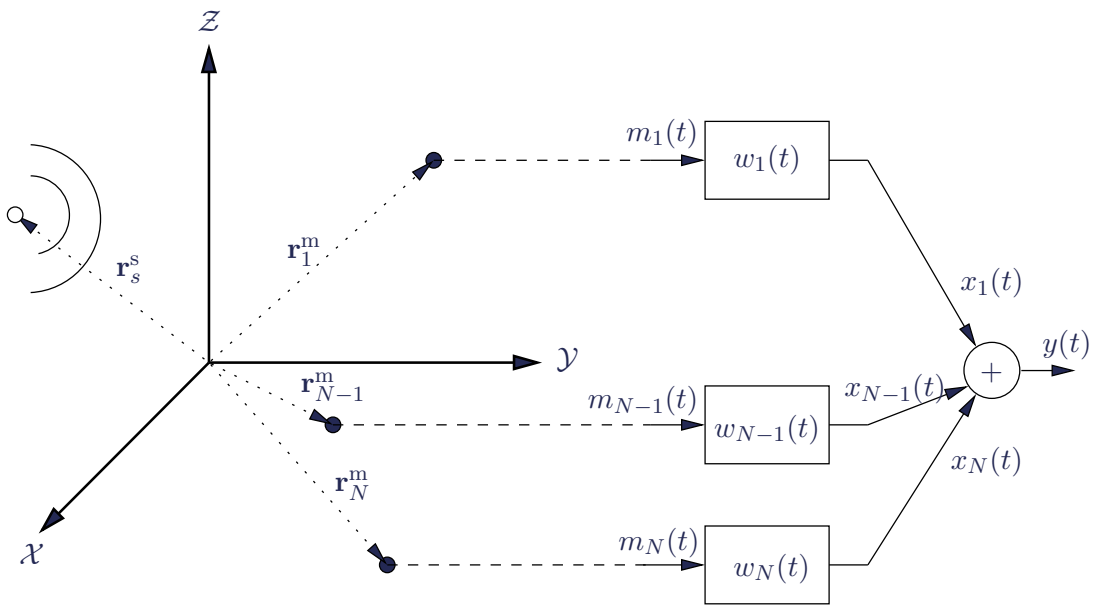
- **Extraction**

The array is polarized in some direction of interest, where a sound source is present. The signal at the array output is expected to mainly reproduce the focused sound source, while all other sounds are filtered out.



Formalization

Architecture: the N microhones outputs $m_n(t)$ are first processed with some adequate linear filters with impulse response $w_n(t)$, and their outputs are then combined to form the beamformer output $y(t)$.



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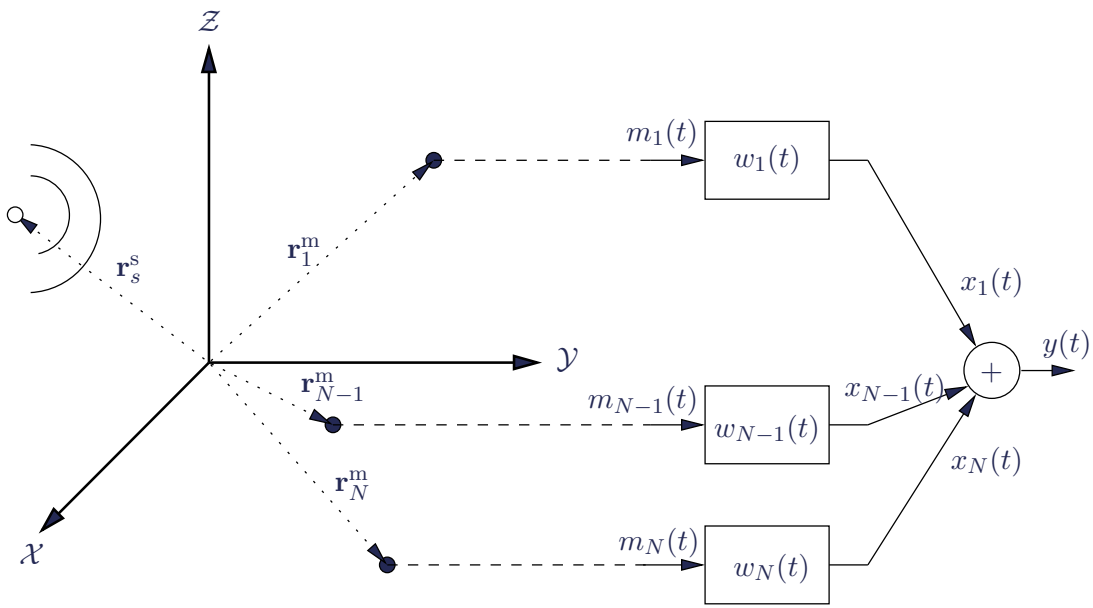

One can then write

$$y(t) = \sum_{n=1}^{N} w_n(t) * m_n(t) \tag{41}$$

Formalization

We saw earlier that

$$M_n(k) = \sum_{s=1}^{S} V_n(\mathrm{r}_s^{\mathrm{s}}, k) S_s^0(k) + B_n(k),$$

So we can write, in the frequency domain

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$$Y(k) = \sum_{s=1}^{S} D(\mathrm{r}_s^{\mathrm{s}}, k) S_s^0(k) + \sum_{n=1}^{N} W_n(k) B_n(k), \tag{42}$$

where $W_n(k)$ is the frequency response of the n^{th} filter placed after the n^{th} microphone, and $D(\mathrm{r}, k)$ the beampattern evaluated at source position $\mathrm{r}_s^{\mathrm{s}}$, with

$$D(\mathrm{r}, k) = \sum_{n=1}^{N} W_n(k) V_n(\mathrm{r}, k). \tag{43}$$

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$$M_n(k) = \sum_{s=1}^{S} V_n(\mathrm{r}_s^{\mathrm{S}}, k) S_s^0(k) + B_n(k),$$

So we can write, in the frequency domain

$$Y(k) = \sum_{s=1}^{S} D(\mathrm{r}_s^{\mathrm{S}}, k) S_s^0(k) + \sum_{n=1}^{N} W_n(k) B_n(k), \tag{42}$$

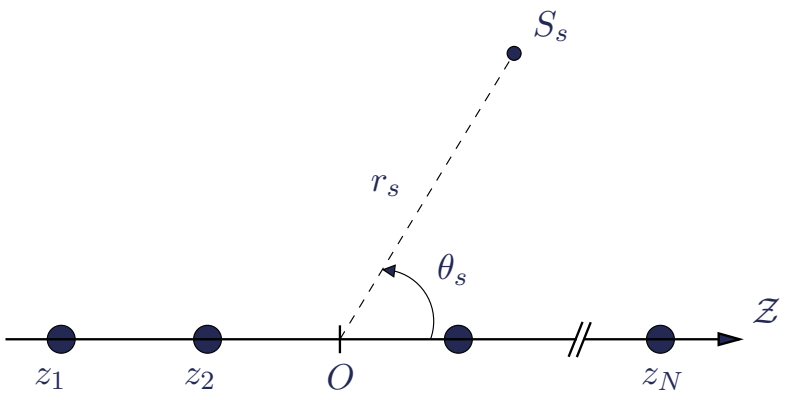
where $W_n(k)$ is the frequency response of the n^{th} filter placed after the n^{th} microphone, and $D(\mathrm{r}, k)$ the beampattern evaluated at source position $\mathrm{r}_s^{\mathrm{S}}$, with

$$D(\mathrm{r}, k) = \sum_{n=1}^{N} W_n(k) V_n(\mathrm{r}, k). \tag{43}$$

Again, if one considers the front waves case, one can define the **farfield beampattern** D^∞ as $D^\infty(\theta, \Psi, k) = \lim_{r \to +\infty} D(\mathrm{r}, k)$

For a linear microphones array and planar waves

Lets consider the following array geometry (already seen in Sec. 1):



We saw earlier:

$$V_n^{\infty}(\theta, \Psi, k) = \lim_{r \to +\infty} V_n(\mathrm{r}, k),$$

and for a linear array

$$V_n^{\infty}(\theta, k) = e^{-jkz_n \cos\theta}.$$

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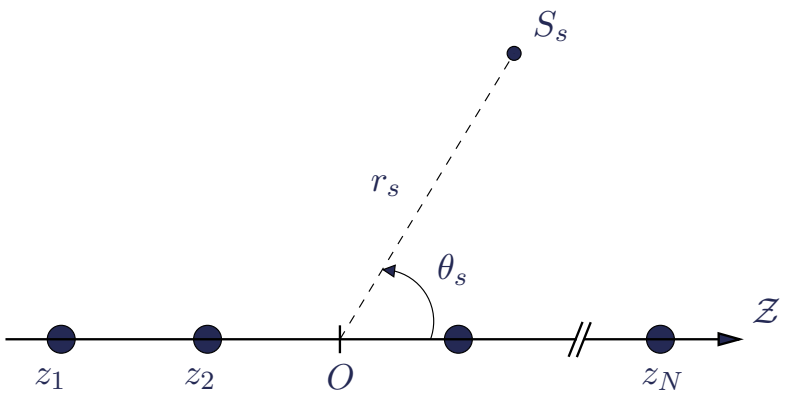

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$$V_n^{\infty}(\theta, k) = e^{-jkz_n \cos\theta}.$$

Then, the array beampattern of a linear microphones array writes as

$$D^{\infty}(\theta, k) = \sum_{n=1}^{N} W_n(k) e^{-jkz_n \cos\theta} \tag{44}$$

For a linear microphones array and planar waves

In the following, we shall consider a linear array with N microphones evenly placed along the $\mathcal{Z}$ axis at abscissa z_n such that

$$z_n = \left(n - \frac{N+1}{2} \right) d, \tag{45}$$

with d the interspace between two successive microphones. Then, the array size is $L = (N-1)d$.

For a linear microphones array and planar waves

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with d the interspace between two successive microphones. Then, the array size is $L = (N-1)d$.

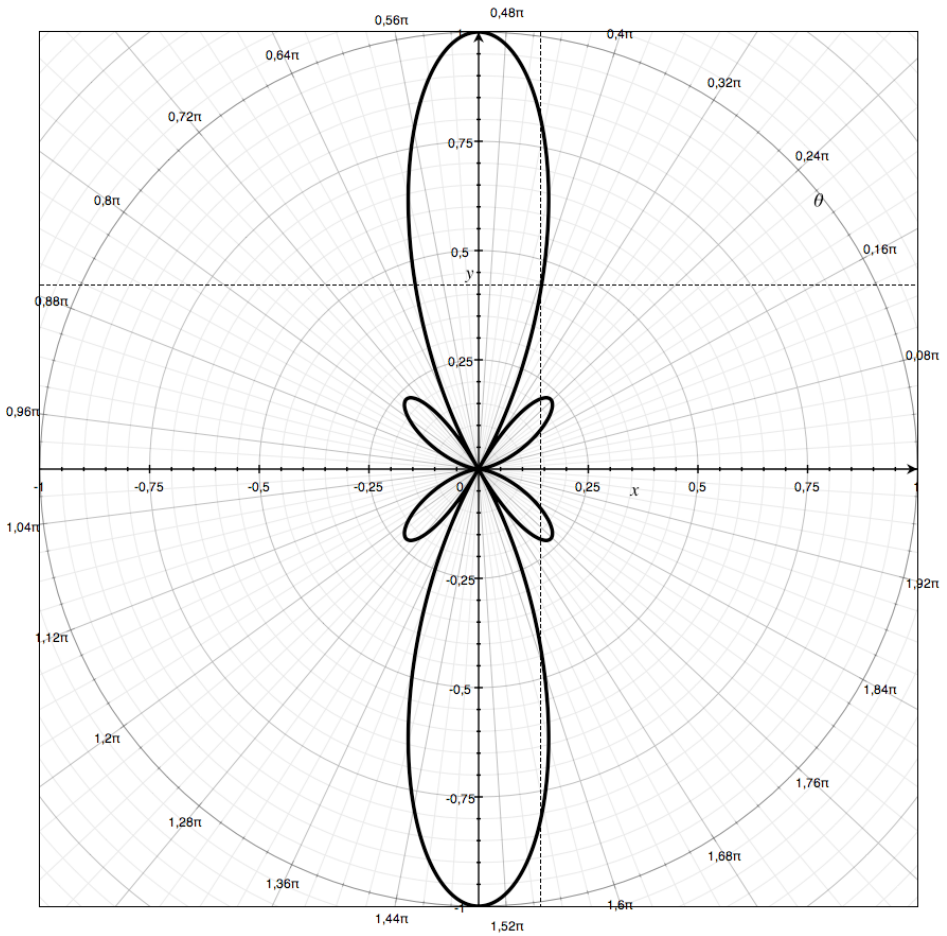
For now, lets suppose that each filter of the beamformer has the same frequency response $W_n(k) = 1$. One then have

$$D^{\infty}(\theta, k) = \sum_{n=1}^{N} e^{-jk\left(n - \frac{N+1}{2}\right) d\cos\theta} = \frac{\sin\left(k\frac{Nd}{2}\cos\theta\right)}{\sin\left(k\frac{d}{2}\cos\theta\right)} \tag{46}$$

$$= \frac{\sin\left(\frac{\pi f}{c} Nd\cos\theta\right)}{\sin\left(\frac{\pi f}{c} d\cos\theta\right)}. \tag{47}$$

For a linear microphones array and planar waves

We can now represent the farfield beampattern $D^{\infty}(\theta, k) = \frac{\sin\left(\frac{\pi f}{c} Nd \cos\theta\right)}{\sin\left(\frac{\pi f}{c} d \cos\theta\right)}$:



- The array pattern is symetric w.r.t. the array axis;

Plot obtained for $N = 8$, $f = 1500Hz$, $d = 5.6cm$, so that $L = 39.6cm$.

For a linear microphones array and planar waves

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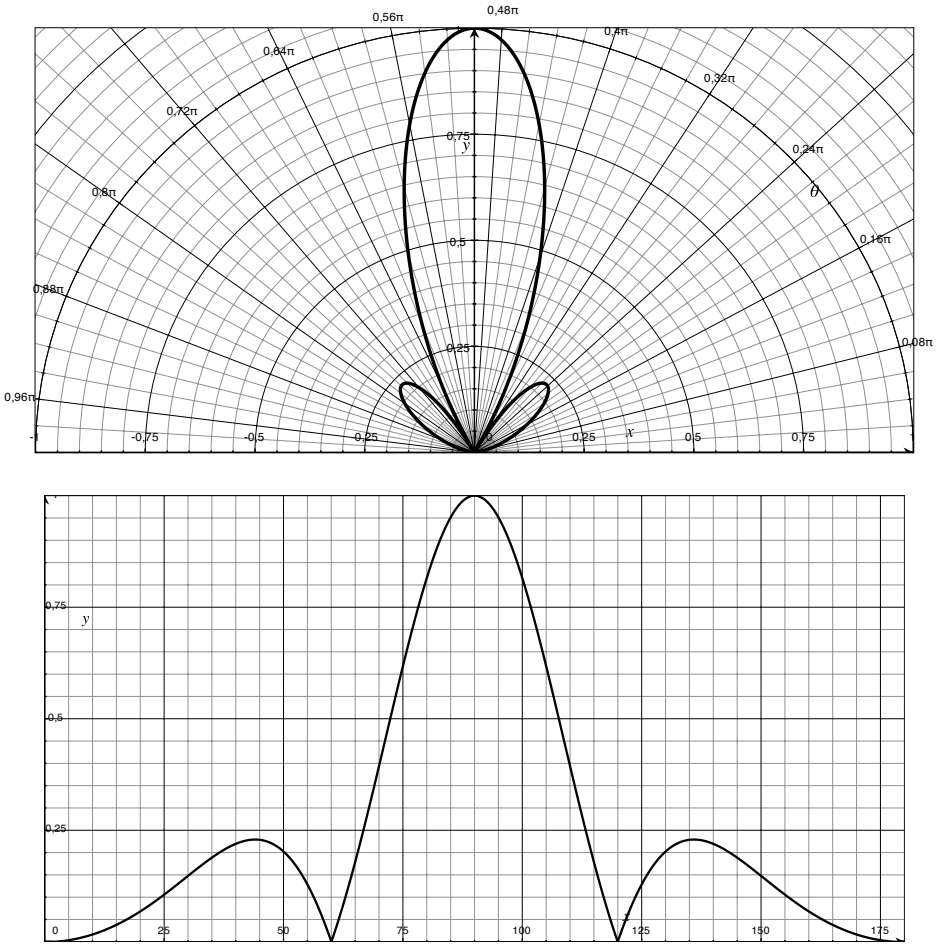

- The array pattern is symetric w.r.t. the array axis;
- Choosing $W_n(k) = 1$ makes the array pointing towards the frontal direction $\theta = \pi/2$;

Plot obtained for $N = 8$, $f = 1500Hz$,
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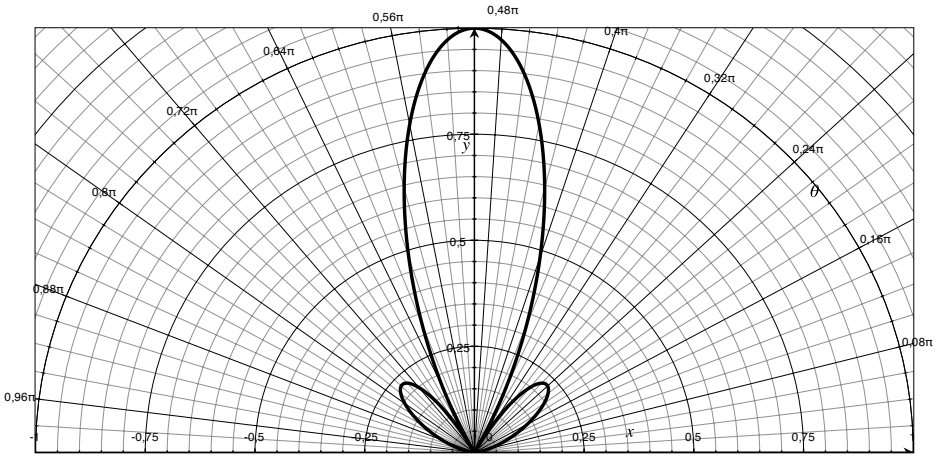

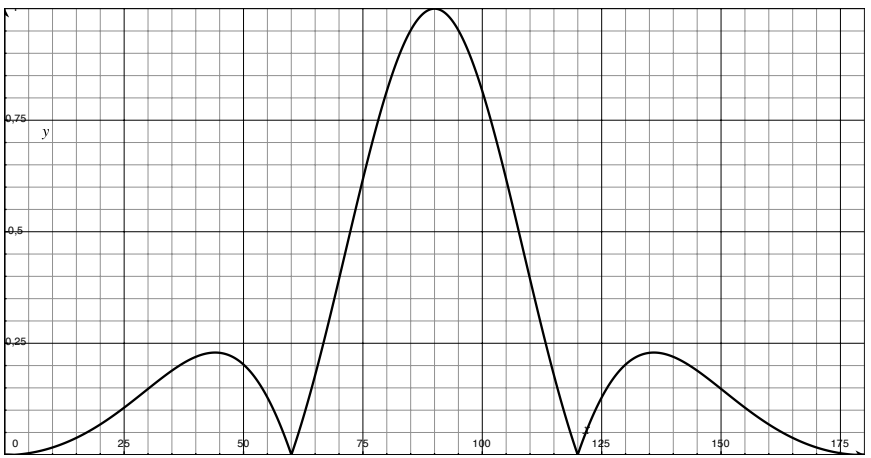

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- The array pattern is symetric w.r.t. the array axis;
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- The main lobe is significantly wide ...

For a linear microphones array and planar waves

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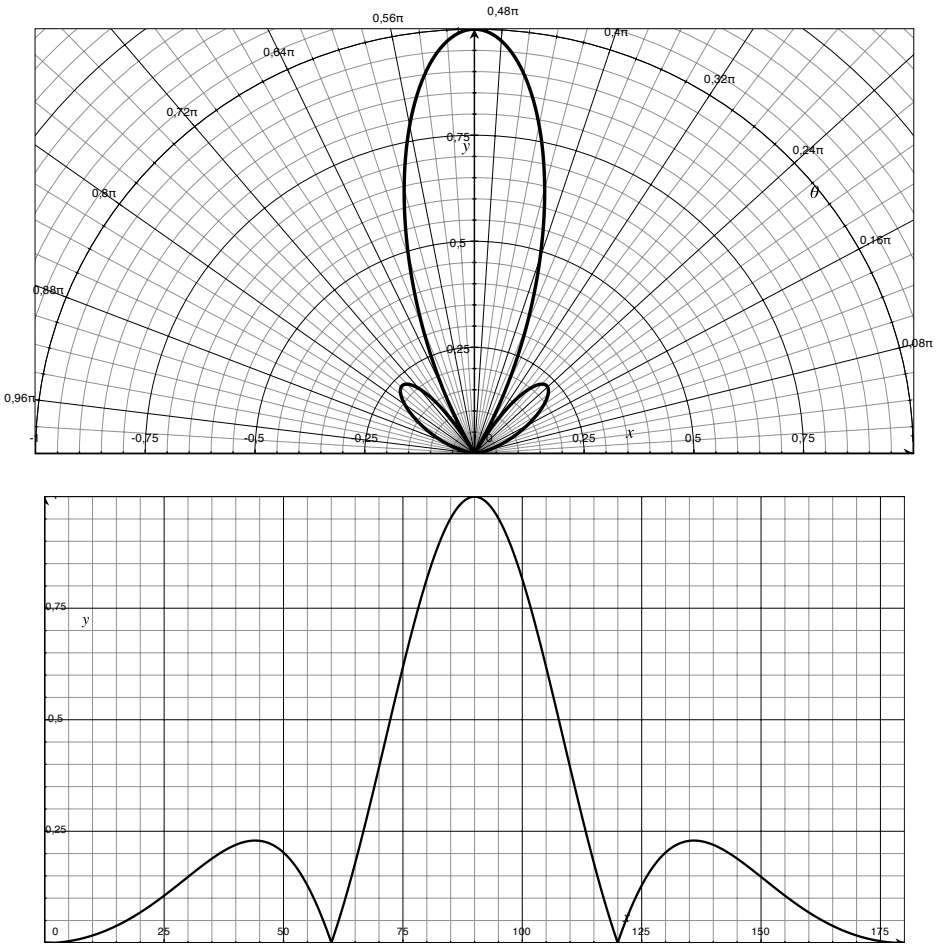

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- ... and secondary lobes also appear in the beampattern.

$\rightarrow$ the spatial filtering is not perfect!

For a linear microphones array and planar waves

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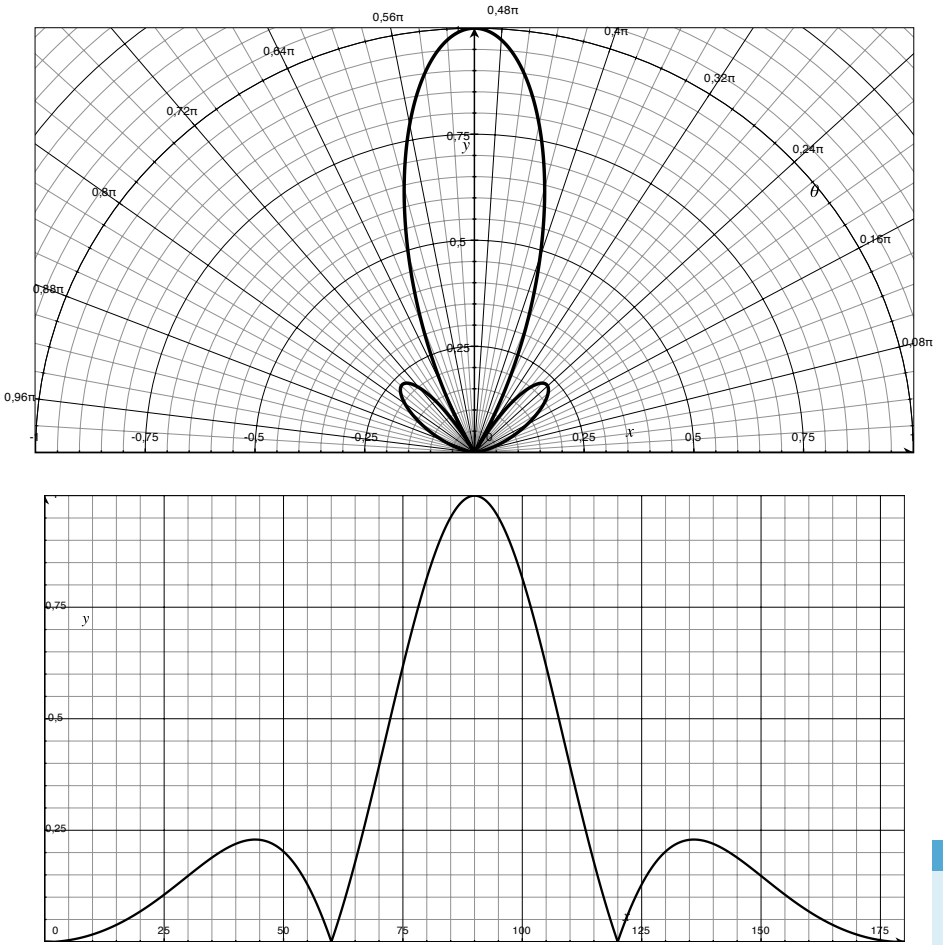

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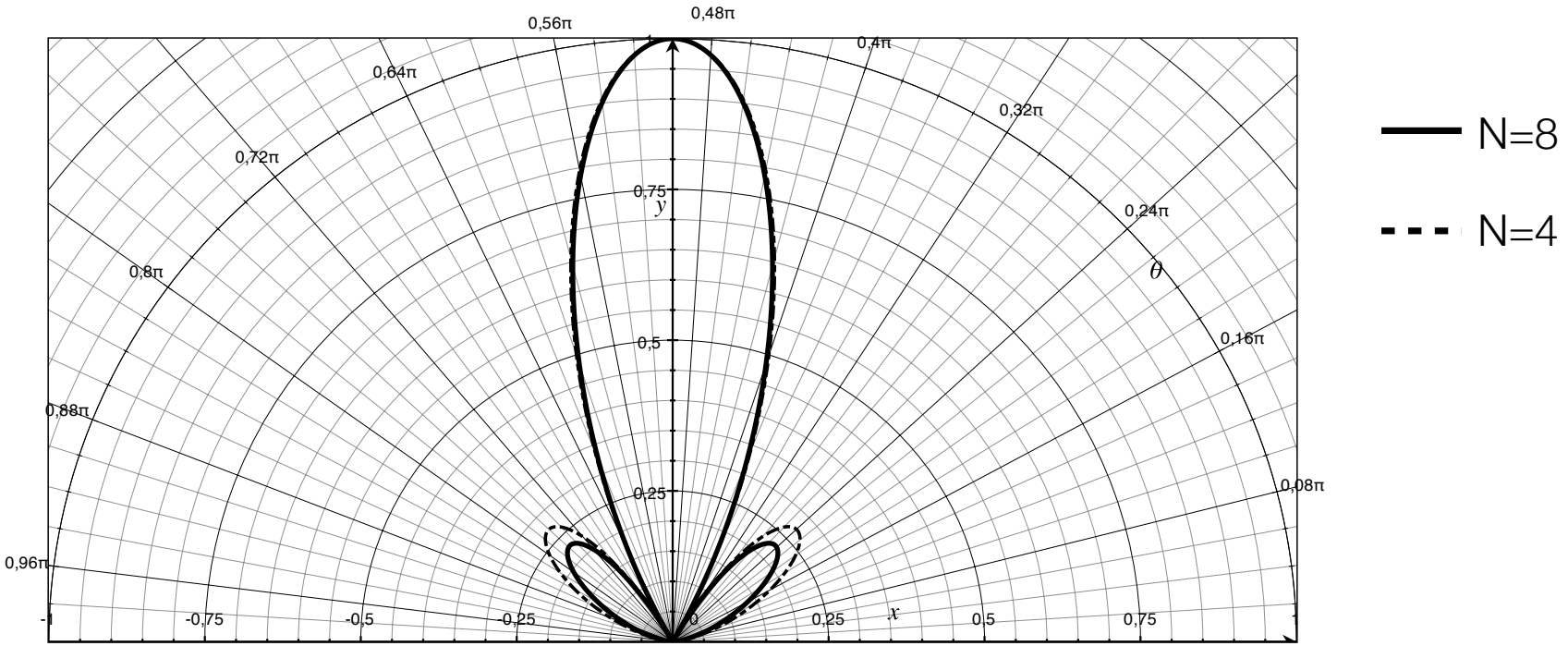
- ... and secondary lobes also appear in the beampattern.

$\rightarrow$ the spatial filtering is not perfect!

What is the influence of the array geometry (N, L parameters) and of the frequency f?

Influence of the array geometry and frequency on the beampattern

→ Array size $L = 39.6$cm is constant, and $f = 1500$Hz.

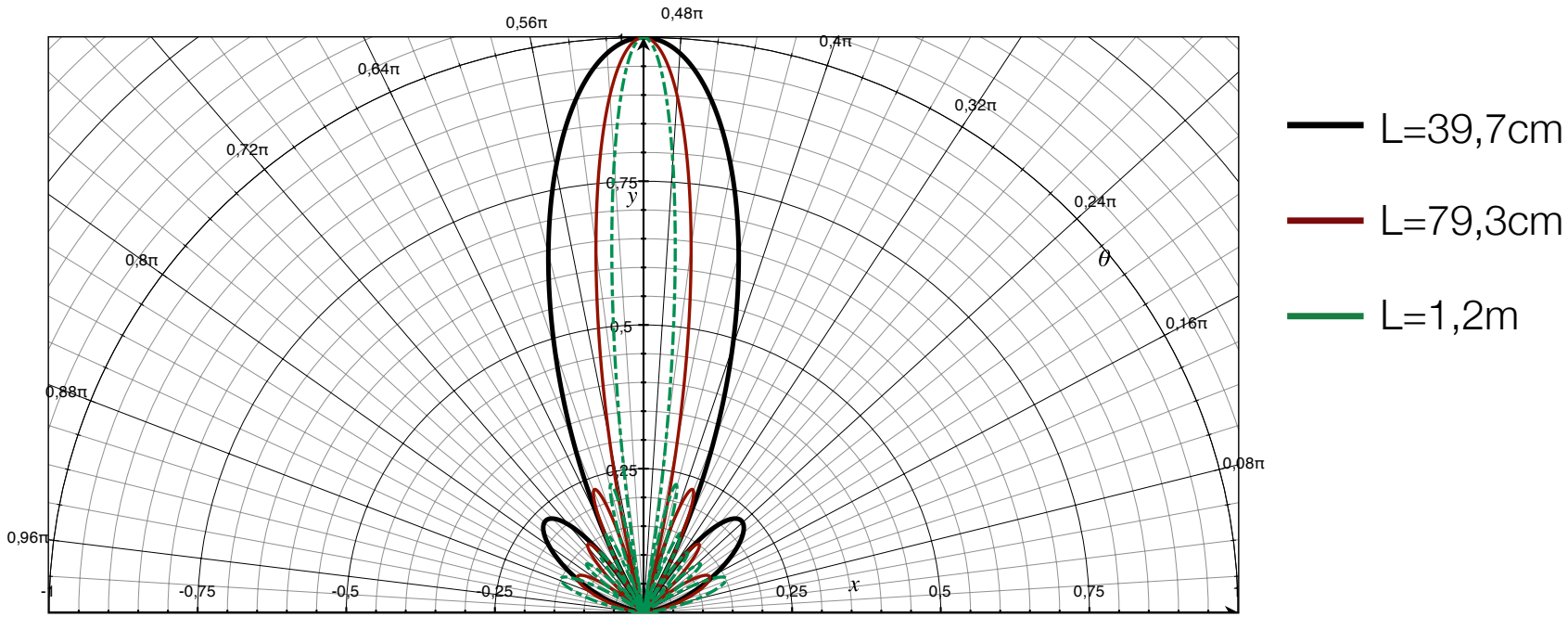


Influence of N

Sidelobes level is reduced (for a constant array length).

Influence of the array geometry and frequency on the beampattern

$\rightarrow$ $N = 8$ microphones, and $f = 1500$Hz.

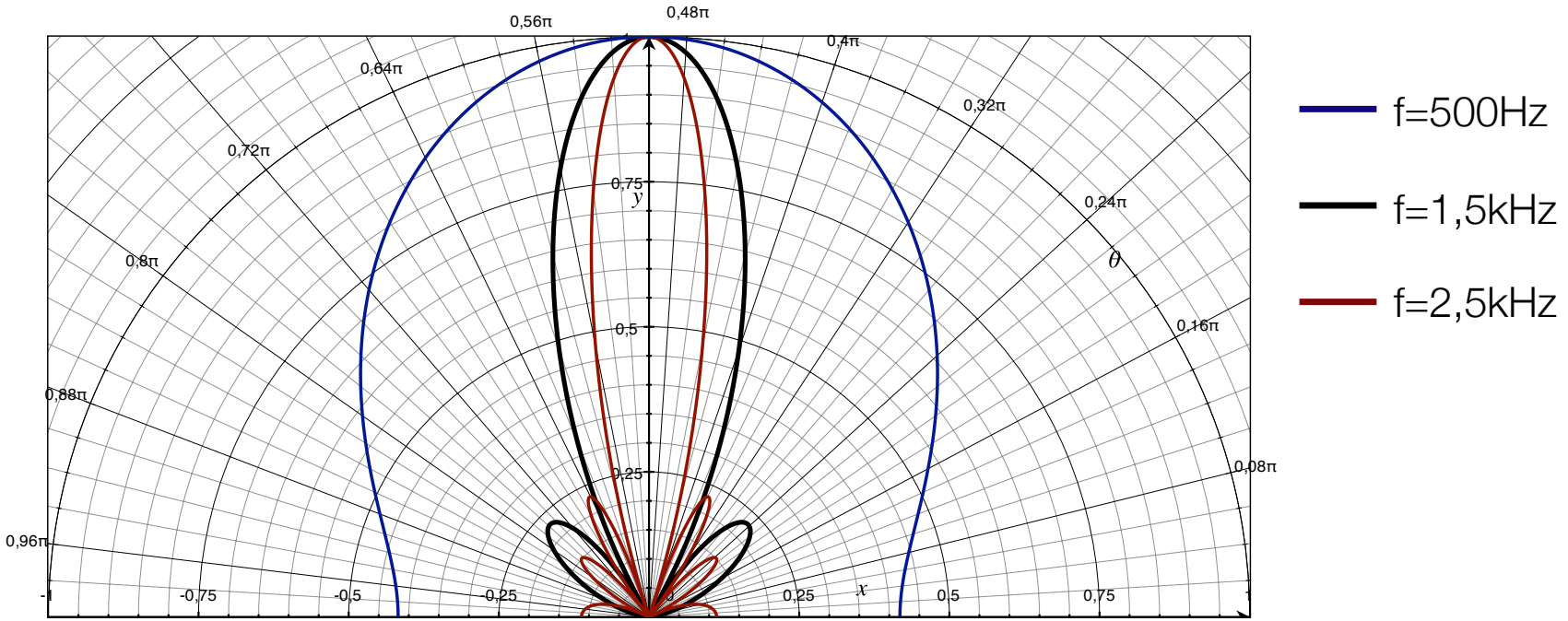


Influence of L

The spatial filtering is more selective for a wide array.

Influence of the array geometry and frequency on the beampattern

$\rightarrow$ $N = 8$ microphones, and $L = 39.6$cm.

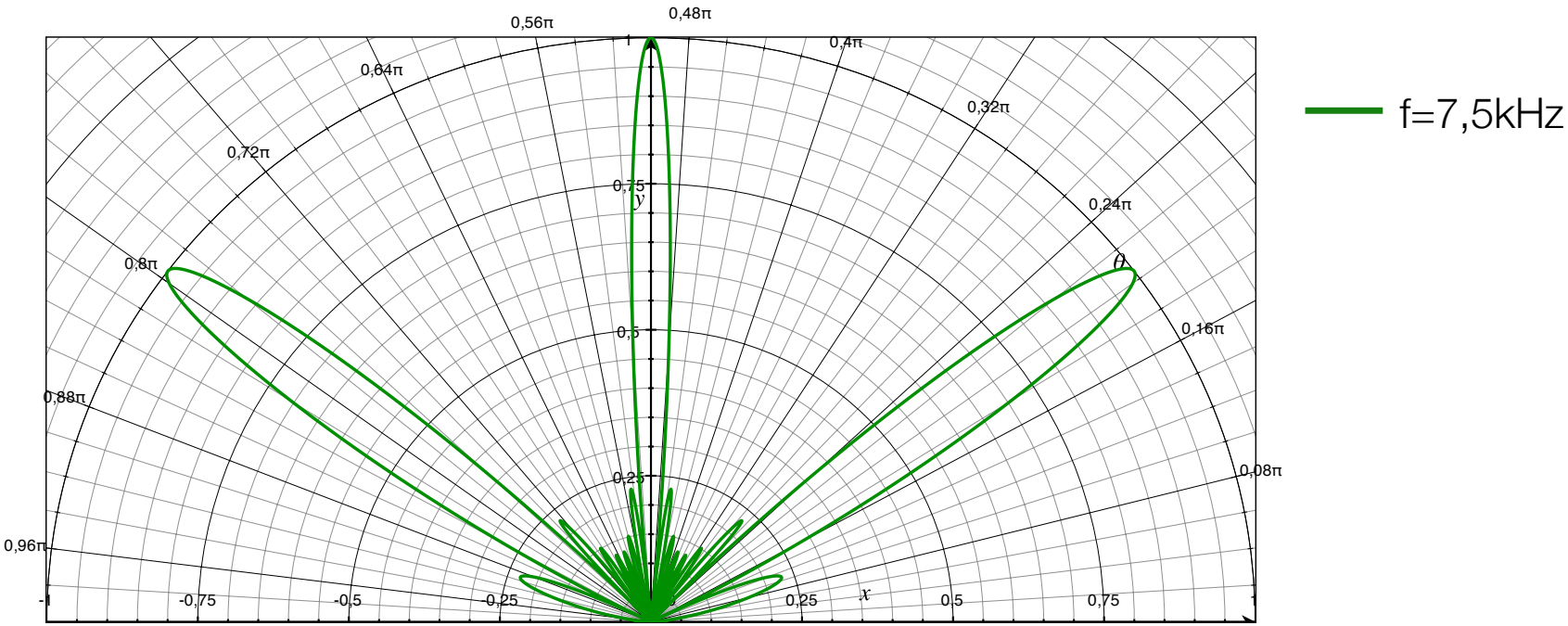


Influence of f

The spatial selectivity is (far) lower for low frequencies.

Influence of the array geometry and frequency on the beampattern

$\rightarrow$ $N = 8$ microphones, and $L = 39.6$cm.

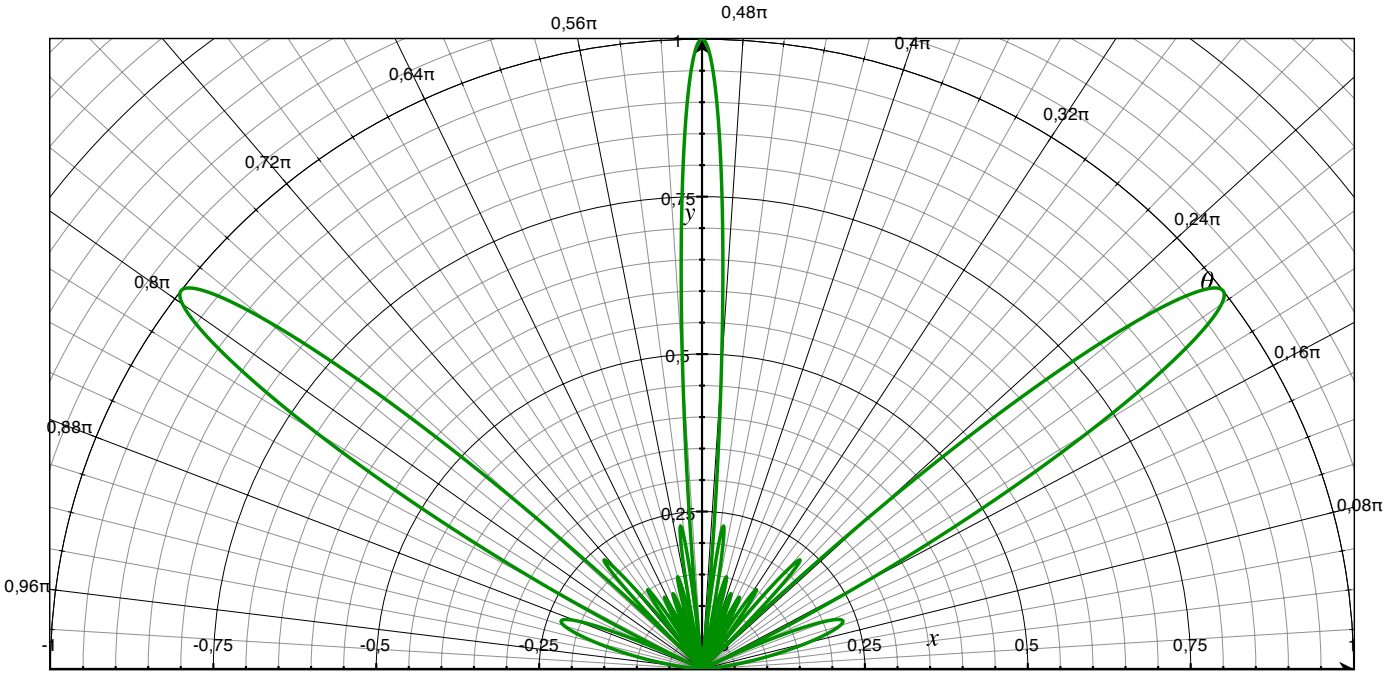


Influence of f

Be careful with spatial aliasing!

Influence of the array geometry and frequency on the beampattern

$\rightarrow$ $N = 8$ microphones, and $L = 39.6$cm.

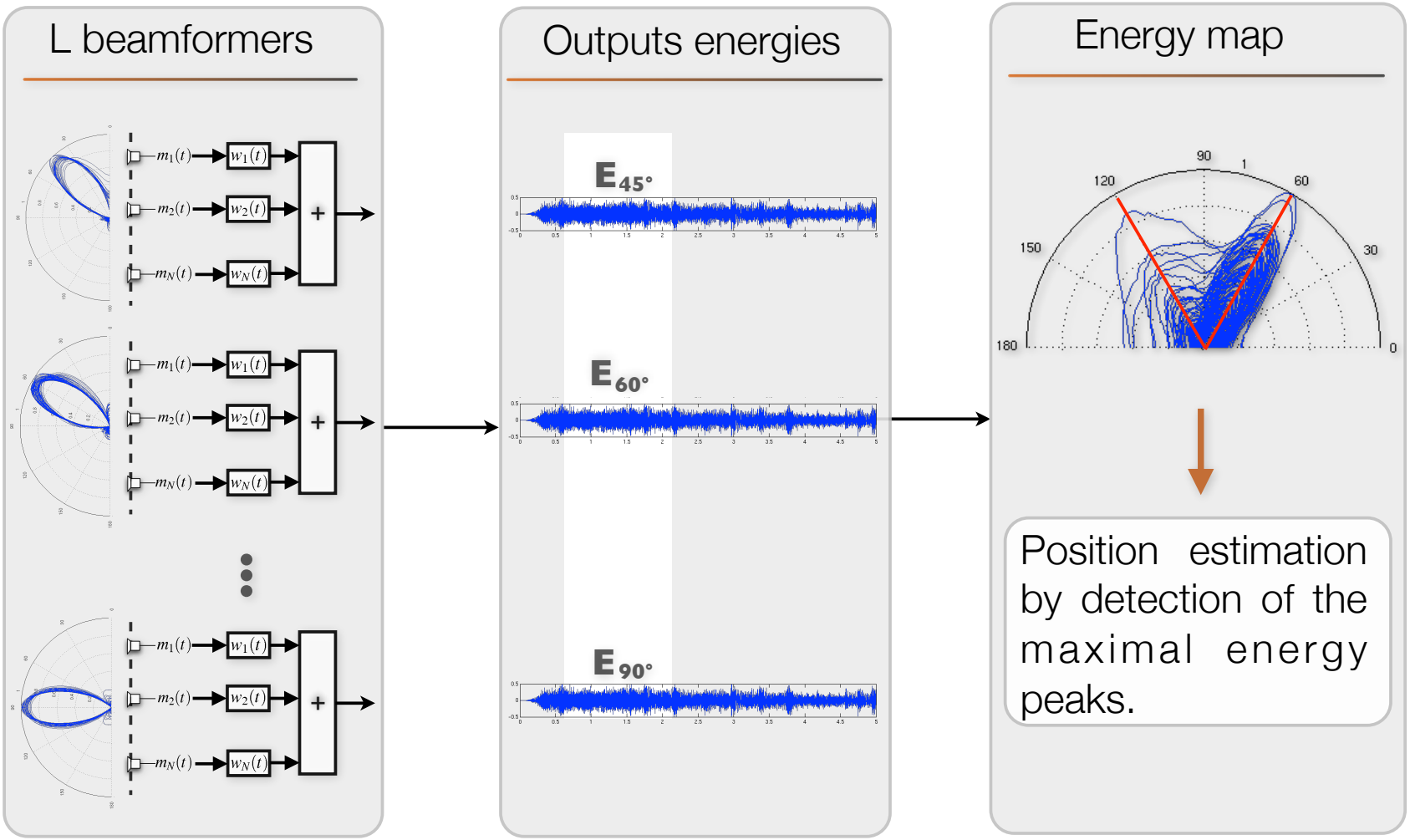


- array size $\leftrightarrow$ observation window of a signal
- microphone interspace d $\leftrightarrow$ sampling time for a discrete signal

Sampling theorem

$$d < d_{\text{max}} = \frac{\lambda_{\text{min}}}{2} = \frac{c}{2f_{\text{max}}}. \tag{48}$$

Using a "traditional" beamformer for localization



→ **How can we polarize an array in such directions?**

Using a "traditional" beamformer for localization

Using filters $W_n(k) = 1$ made an array mainly sensitive to frontal positions: **why is that?**

$\rightarrow$ it allows a *constructive* sum of the filters outputs only when a source is coming from the front, i.e. only when there is **no delay** between all microphones outputs.

To polarize the (linear) array towards the direction θ_0, one needs to compensate for the delays caused by the propagation from direction θ_0. Then:

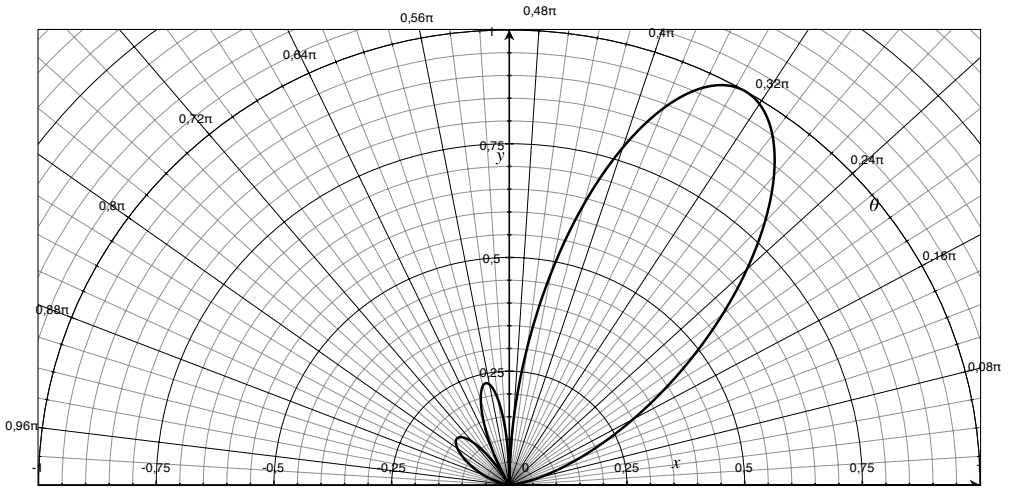
$$W_n(k) = e^{jkz_N \cos\theta}, \tag{49}$$

which defines a pure phase shifter. The farfield beampattern, for an evenly spaced linear array, then writes as

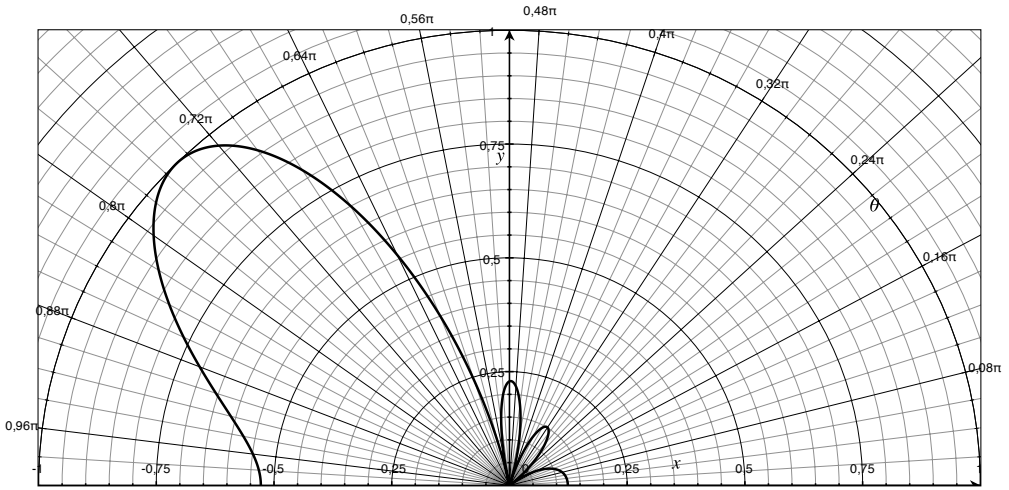
$$D^{\infty}(\theta) = \frac{\sin\left(\frac{\pi f}{c} Nd(\cos\theta_0 - \cos\theta)\right)}{\sin\left(\frac{\pi f}{c} d(\cos\theta_0 - \cos\theta)\right)} \tag{50}$$

Using a "traditional" beamformer for localization

Some examples, for $N = 8$ microphones array of length $L = 39.6$cm at frequency $f = 1500$Hz:



$\theta_0 = 60°$



$\theta_0 = 135°$

Beamforming techniques: conclusion

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Pros:

- simple approach, only relying on linear filters placed at the output of each microphone
- limited computational requirements, well-adapted to low-powered / specialized hardware (FPGA)
- can easily be used in real time
- can also be used for sound source extraction (naive approach)

Cons:

- the spatial filtering is really effective only for very wide arrays
- spatial filtering performances very poor for low frequencies
- generally requires a high number of microphones

(See Van Trees, 2002 for (far!) more details).

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